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ON THE NATURE OF COMPETITION WITH DIFFERENTIATED PRODUCTS

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There is an old debate going back to Bertrand (1883) and Edgeworth (1925) about the fragility of market equilibrium when competition arises among a few sellers. It was Hotelling's belief that price instability vanishes when products are differentiated (Hotelling, 1929). Recently attention was drawn to the fact that, even if product differentiation can weaken the forces leading to price instability, it cannot eliminate completely the possibility of price cycles (d'Aspremont *et al.* 1979). The purpose of this paper is to show that *vertical* and *horizontal* product differentiation do not operate in the same manner: a stable market outcome (including an endogenous product specification) arises more frequently with the former than with the latter.[1]

Horizontal product differentiation is rooted in taste differences. More precisely, the potential customers have *heterogeneous* preferences about the proportion in which the attributes of the product should be combined. A wide range of substitute products can then survive in the same market simply because each of them combines the various attributes of the product in a proportion suitable to a particular segment of customers: between two products in the range the level of some attributes is augmented while that of others is lowered. Each variety has its own circle of customers, exactly as the inhabitants located around a particular shop form its potential market. None the less, competitors can raid these privileged market shares by adequate price cuts.

By contrast, vertical product differentiation refers to a class of products which cohabit simultaneously on a given market, even though customers agree on a unanimous ranking between them: between two products in the range the level of all attributes is augmented or lowered. The survival of a low-quality product then rests on the seller's ability to sell it at a reduced price, compensating thereby for the higher *a priori* attractiveness of a more desirable quality. The seller of a low-quality product will specialise in the segment of customers whose propensity to spend on the corresponding range of products is low, either because they have relatively lower income, or relatively less intensive preferences, than other customers. At the same time, the seller of a high-quality product will enjoy an absolute advantage over his competitor.[2]

Interestingly, it turns out that price and product competition does not lead to the same results in these two situations, regarding the stability of the market equilibrium. To stress the differences, we have chosen to concentrate on two

[1] Another distinguishing feature of horizontal and vertical product differentiation, not discussed here, is related to the market structure that may emerge under these alternative conditions (see Shaked and Sutton (1985) for a detailed discussion).

[2] To the best of our knowledge, the distinction between vertical and horizontal product differentiation is due to Lancaster (1979).

simple models which capture the essential features of horizontal versus vertical product differentiation. Both of them are borrowed from the paradigm of location theory.

I. THE MODELS

Let us consider a road along a closed valley and a population spread uniformly along that road between point 0 (the endpoint of the valley) and point 1. In the first example depicted in Fig. 1, two shops selling a homogeneous product are established inside the residential area, represented by the interval [0, 1], at s_1 (seller 1) and $s_2 \geqslant s_1$ (seller 2) with $s_1, s_2 \in S_{\mathrm{I}} \underset{\mathrm{Def}}{=} [0, 1]$. In the second example, represented in Fig. 2, shops are not allowed inside the residential area because of zoning regulations, but the shopkeepers still want a site on the road. Obeying this constraint, seller 1 has set up his shop at $s_1 \geqslant 1$ and seller 2 at $s_2 \geqslant s_1$, with $s_1, s_2 \in S_{\mathrm{II}} \underset{\mathrm{Def}}{=} [1, \infty[$.

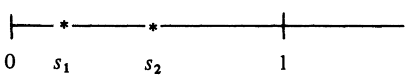

Fig. 1. Example I.

Fig. 2. Example II.

The first example corresponds exactly to Hotelling's 'Main Street' case, and constitutes the prototype of horizontal product differentiation (call it 'model I'). If the customers incur a transportation cost when moving from their home to either shop, those located closer to seller 1 than seller 2 prefer the product at shop s_1 than the same product at shop s_2, and vice versa (here the distance is the only attribute which distinguishes the products).

Consider now the second example. It is clear that all inhabitants now prefer the product at the shop located at s_1 to the same product available at the shop located at s_2: for any consumer the distance from his residence to s_1 is shorter than that to s_2, so that transportation costs are lower. There is a unanimous agreement among consumers that seller 1 has a better product than seller 2. This example will be our prototype of vertical product differentiation (call it 'model II').

To compare the nature of competition between the sellers in model I (horizontal differentiation) and model II (vertical differentiation) respectively, we assume that, in both cases, the transportation costs $t(s, s')$ between locations s and s' are derived from a quadratic utility function and given by

$$t(s, s') \underset{\mathrm{Def}}{=} c|s-s'| + d(s-s')^2, \quad c > 0, \quad d > 0.$$ [1]

[1] The analysis of model I under the alternative assumptions $c > 0$, $d = 0$, or $c = 0$, $d > 0$ has been conducted elsewhere (d'Aspremont *et al.* 1979).

We also assume that each customer consumes a single unit of the product irrespective of its price and that the product is produced at zero cost. In what follows, we first compare price competition with fixed locations, using the concept of a non-cooperative price equilibrium. Afterwards, we study the problem of spatial (or product) competition, leading to the selection of a certain position of firms in the geographical space; here we use the non-cooperative concept of perfect equilibrium.

II. THE RESULTS

II. 1. *Price competition*

To start out, let us consider price competition in model I. We assume that both merchants are symmetrically located, as in Fig. 3, as $s_1 = \frac{1}{2} - a$ and $s_2 = \frac{1}{2} + a$

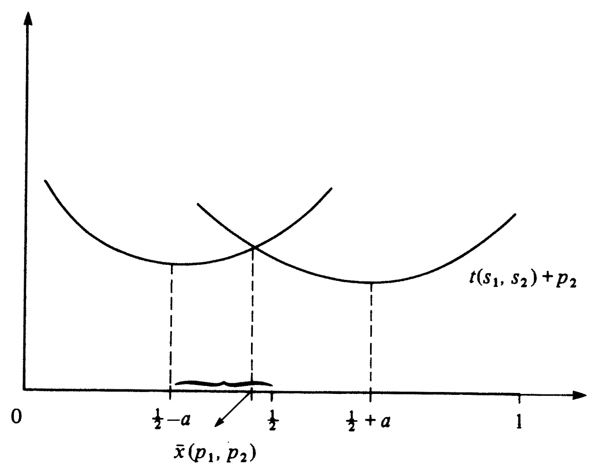

Fig. 3. Market areas in Model I.

respectively, with $0 \leqslant a \leqslant \frac{1}{2}$.[1] Denote by p_1 (resp. p_2) the mill price quoted by seller 1 (resp. seller 2). Since the product is homogeneous, a customer buys from the seller with the lowest delivered price, namely mill price plus transportation cost. The use of this choice rule enables us to derive the demand addressed to each firm, conditionally upon the price charged by its competitor. Let $\bar{x}(p_1, p_2)$ be the location of the marginal consumer, where $\bar{x}(p_1, p_2)$ obtains as the solution to the equation

$$p_1 + c|s_1 - x| + d(s_1 - x)^2 = p_2 + c|s_2 - x| + d(s_2 - x)^2.$$

Customers between 0 and $\bar{x}(p_1, p_2)$ are served by the first seller, while those located between $\bar{x}(p_1, p_2)$ and 1 are served by the second seller. Depending upon the position of $\bar{x}(p_1, p_2)$ relative to 0, s_1, s_2 and 1, the demand function is piecewise linear with five different price domains (see Fig. 3 where $s_1 < \bar{x}(p_1, p_2) < s_2$).

[1] For reasons to become clear later, there is no need for our purpose to investigate price competition in the nonsymmetric case.

As an example we give the demand function addressed to firm 1, given the price $\bar{p}_2$ announced by firm 2:

$$\mu_1(p_1, \bar{p}_2) = 0, \quad \text{if} \quad p_1 \geqslant p'_1 \underset{\text{Def}}{=} \bar{p}_2 + 2ac + 2ad;$$

$$= \frac{\bar{p}_2 - p_1 + 2ac + 2ad}{4ad}, \quad \text{if} \quad p'_1 > p_1 \geqslant p''_1 \underset{\text{Def}}{=} \bar{p}_2 + 2ac + 4a^2d;$$

$$= \frac{\bar{p}_2 - p_1 + c + 2ad}{4ad + 2c}, \quad \text{if} \quad p''_1 > p_1 \geqslant p'''_1 \underset{\text{Def}}{=} \bar{p}_2 - 2ac - 4a^2d;$$

$$= \frac{\bar{p}_2 - p_1 + 2ad - 2ac}{4ad}, \quad \text{if} \quad p'''_1 > p_1 \geqslant p''''_1 \underset{\text{Def}}{=} \bar{p}_2 - 2ac - 2ad;$$

$$= 1, \quad \text{if} \quad p''''_1 > p_1 \geqslant 0.$$

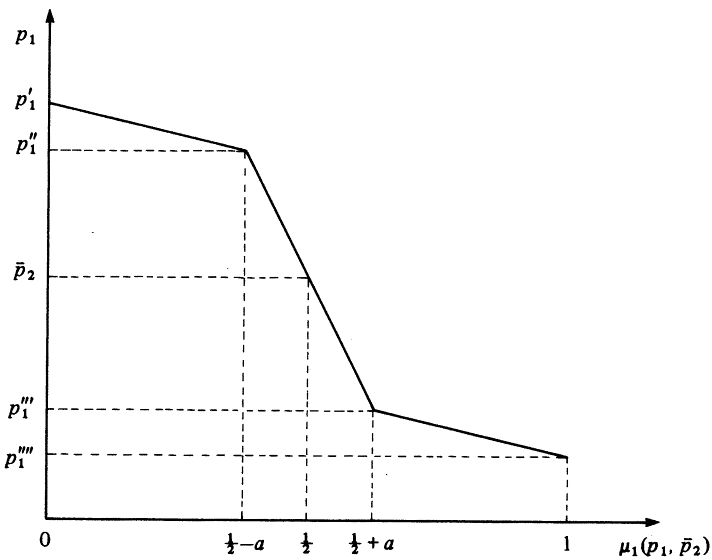

Fig. 4. Firm 1's demand in Model I.

Notice that, at the price $p_1 = p'''_1$, when the market boundary is exactly at $\bar{s}_1 = \frac{1}{2} + a$, the demand function exhibits a kink destroying its concavity. A similar analysis leads to the demand function addressed to firm 2. This demand function shares the same characteristics: continuity, piecewise linearity and absence of concavity. Define $P_i(p_i, p_j) = p_i \mu_i(p_i, p_j)$ as the profit function of firm i, $i = 1, 2$. A non-cooperative *price equilibrium* is a pair of prices (p_1^*, p_2^*) such that, given the price of its competitor, no firm can raise its profits by changing its price unilaterally.

It is shown in Proposition 1 of the Appendix that a price equilibrium ceases to exist as soon as the distance $2a$ between the two firms is small enough.[1] Intuitively, the argument is as follows. The pattern of the profit function depends both on $\bar{p}_2$ and on the parametric values a (which measures the distance between

[1] Interestingly, here is an example where existence of an equilibrium fails even though payoffs are continuous. This invalidates the widespread opinion according to which non-existence in location models would be due to discontinuities.

the two firms) and c and d (which measure the relative weight of the quadratic and linear terms in the transport cost function respectively). Should a price equilibrium exist, the natural candidate for fulfilling the role is the pair (p_1^*, p_2^*), where p_1^* is the best reply against p_2^* in the domain where the market boundary is in between the two firms, and vice versa. (Indeed, we should not expect equilibrium strategies elsewhere, since that would imply that one of the two firms does not completely supply its own hinterland. In this case, that firm would have an incentive to lower its price so as to recover it, contradicting the fact that such a pair is a price equilibrium.) To ascertain that the pair (p_1^*, p_2^*) is a price equilibrium, one must further exclude the possibility that a firm can increase its profits by charging a price which would capture customers located beyond his competitor's shop. Actually, it is always profitable for a firm to do so when competitors are close enough (see the Appendix).

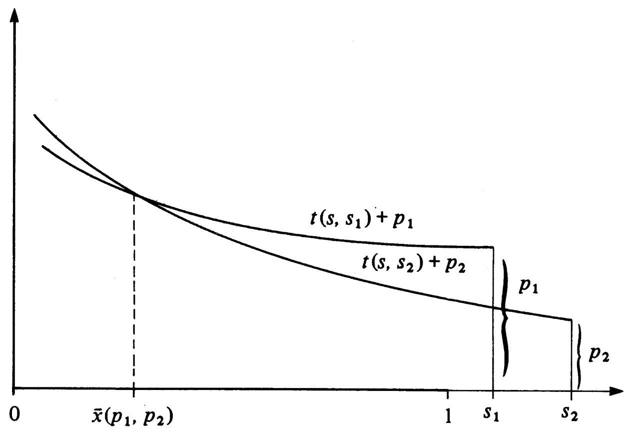

Fig. 5. Market areas in Model II.

Thus, in model I, there are pairs of locations for which no price equilibrium exists.[1] We shall use this property when comparing the nature of spatial competition in the horizontal – versus vertical – differentiation context.

Let us now turn to the analysis of price competition in model II. Given prices p_1 and p_2, customers located between 0 and $\bar{x}(p_1, p_2)$ are served by the first merchant while those located between $\bar{x}(p_1, p_2)$ and 1 are served by the second merchant (see Fig. 5).

The market boundary $\bar{x}$ obtains as the solution of the equation

$$p_1 + c(s_1 - x) + d(s_1 - x)^2 = p_2 + c(s_2 - x) + d(s_2 - x)^2,$$

[1] It was suggested by Eaton and Lipsey (1978) that a way of solving the existence problem is to prevent the firms from using strategies which completely eliminate their competitors. When the definition of a price equilibrium is amended on account of this restriction, one obtains the so-called 'modified zero conjectural variation' (ZCV) price equilibrium. It is indeed true that such an equilibrium always exists in the case of linear transportation cost functions. However, it can be shown that, for some pairs of locations, a modified ZCV price equilibrium does not exist in the case of linear-quadratic transport costs for exactly the same reason as in the case of a price equilibrium, i.e. profit functions are not quasi-concave over the restricted domain of price.

i.e. $$\bar{x}(p_1,p_2) = \frac{p_2-p_1+c(s_2-s_1)+d(s_2^2-s_1^2)}{2d(s_2-s_1)},$$

so that the demand functions to the two firms are, respectively, given by

$$\begin{aligned}
\mu_1(p_1,p_2) &= 0, \quad \text{if} \quad p_1 \geqslant p_2+c(s_2-s_1)+d(s_2^2-s_1^2) \underset{\text{Def}}{=} p_1';\\
&= \frac{p_2-p_1}{2d(s_2-s_1)}+\frac{c+d(s_2+s_1)}{2d}, \quad \text{if} \quad p_1' > p_1 \geqslant p_1'' \underset{\text{Def}}{=} p_2\\
&\qquad + (c-2d)(s_2-s_1)+d(s_2^2-s_1^2);\\
&= 1, \quad \text{if} \quad p_1'' > p_1 \geqslant 0;\\
\mu_2(p_1,p_2) &= 0, \quad \text{if} \quad p_2 \geqslant p_1+(2d-c)(s_2-s_1)-d(s_2^2-s_1^2) \underset{\text{Def}}{=} p_2';\\
&= \frac{p_1-p_2}{2d(s_2-s_1)}+\frac{2d-c-d(s_2+s_1)}{2d}, \quad \text{if} \quad p_2' > p_2 \geqslant p_2'' \underset{\text{Def}}{=} p_1\\
&\qquad -c(s_2-s_1)-d(s_2^2-s_1^2);\\
&= 1, \quad \text{if} \quad p_2'' > p_2 \geqslant 0.
\end{aligned}$$

Defining $P_i(p_i,p_j) = p_i\mu_i(p_i,p_j)$ as the profit function of merchant $i = 1, 2$, we notice that both functions are quasi-concave, ensuring the existence of a price equilibrium, whatever the locations s_1 and s_2 may be, i.e. for any possible vertical differentiation of locations. This property should be contrasted against the price instability we have observed above when differentiation in the locations pattern was horizontal, rather than vertical.

II. 2. *Product Competition*

Following Hotelling (1929), the link between product competition and price decisions is traditionally analysed as a two-stage non-cooperative process, with product decisions (first stage) based upon the belief that ensuing price decisions (second stage) will constitute a price equilibrium. The division in stages is motivated by the very sequential nature of the decision process: the choice of location is prior to the decision on price. Define $[p_1^*(s_1,s_2), p_2^*(s_1,s_2)]$ as a price equilibrium (when it exists!) corresponding to a pair of locations (s_1,s_2). Denote by

$$P_i[s_i,s_j;\, p_i^*(s_i,s_j), p_j^*(s_i,s_j)] = p_i^*(s_i,s_j)\,.\,\mu_i[s_i,s_j;\, p_i^*(s_i,s_j), p_j^*(s_i,s_j)],$$

the profit of firm i when firm i (resp. j) is located at s_i (resp. s_j) and prices are charged at the corresponding price equilibrium. (Clearly, $\mu_i(\ldots)$ stands for the value of the demand function of seller i at the same price equilibrium.) Then a perfect equilibrium is defined as a pair $[(p_1^*,s_1^*), (p_2^*,s_2^*)]$ such that $\forall s_i \in S_{\text{I}}$ (or S_{II}), $i = 1, 2$,

(i) $p_1^* = p_1^*(s_1^*,s_2^*)$ and $p_2^* = p_2^*(s_1^*,s_2^*)$,

(ii) $P_i[s_i^*,s_j^*;\, p_i^*(s_i^*,s_j^*), p_j^*(s_i^*,s_j^*)] \geqslant P_i[s_i,s_j^*;\, p_i^*(s_i,s_j^*), p_j^*(s_i,s_j^*)]$,

for any $s_i \in S_{\text{I}}$ (or S_{II}).[1]

The concept of perfect equilibrium captures the idea that both merchants anticipate, when they choose their locations, the consequences of their choice on

[1] On the concept of perfect equilibrium see Selten (1975).

price competition. In particular they should be aware that this competition will be more severe if they locate close to each other, rather than far apart. On the other hand, if they move 'too' far apart from each other, they weaken their ability of making incursions in the competitor's market. If some compromise exists, which balances these opposite forces, it should stabilise the product choices and resulting prices at a perfect equilibrium.

Consider now product competition via horizontal and vertical product differentiation, respectively. In the prototype of model I, it is trivial to observe that there exists no perfect equilibrium! To be meaningful this concept requires, indeed, that there exists a price equilibrium for any pair of locations (s_1, s_2). However, if the merchants are located sufficiently close to each other, we know that no price equilibrium can exist for the symmetric locations of the sellers. Consequently horizontal product differentiation entails unstable price and product competition between the sellers. Of course, tacit or explicit coordination between them can possibly make it possible to reach an agreement on prices and locations. But as far as the non-cooperative behaviour is considered, no stable price and location settlement should be expected.

What about product competition in model II? It is shown in Proposition 2 of the Appendix that the unique price equilibrium corresponding to the pair of locations (s_1, s_2) is given by

$$p_1^*(s_1, s_2) = (s_2 - s_1)\frac{d(s_1 + s_2 + 2) + c}{3},$$

$$p_2^*(s_1, s_2) = (s_2 - s_1)\frac{d(4 - s_1 - s_2) - c}{3},$$

if $c/d < 4 - s_1 - s_2$, and by

$$p_1^*(s_1, s_2) = (s_2 - s_1)[d(s_1 + s_2 - 2) + c],$$

$$p_2^*(s_1, s_2) = 0,$$

if $c/d \geqslant 4 - s_1 - s_2$.[1] Clearly, if $2d \leqslant c$, the profit $P_2[s_1, s_2; p_1^*(s_1, s_2), p_2^*(s_1, s_2)]$ of seller 2 is in any case equal to zero, while the profit $P_1[s_1, s_2; p_1^*(s_1, s_2), p_2^*(s_1, s_2)]$ is decreasing with s_1, whatever the value of s_2 in S_{II}. In consequence, for any $s_2 \in S_{II}$, the pair $[(s_1^*, p_1^*), (s_2^*, p_2^*)] = (\{1, (s_2 - 1)\ [d(s_2 - 1) + c]\}, (s_2, 0))$ is a perfect equilibrium when $2d \leqslant c$.

On the contrary, when $2d > c$, seller 2 can always choose s_2 large enough so as to verify the condition $c/d < 4 - s_1 - s_2$, guaranteeing himself a strictly positive profit. Furthermore, the profit of seller 1 still decreases with s_1 for any $s_2 \in S_{II}$, so that seller 1 locates at $s_1 = 1$.

The corresponding value of s_2 which maximises the profit of seller 2 then obtains from the first-order condition $d/ds_2\, P_2[1, s_2; p_1^*(1, s_2), p_2^*(1, s_2)] = 0$,

[1] When $c/d \geqslant 4 - s_1 - s_2$, seller 1 uses, at the price equilibrium, a price strategy akin to the limit price strategy used by a monopolist as a barrier to entry. Indeed, p_1^* is the highest price seller 1 can quote, guaranteeing that seller 2 cannot enter the market, even at a zero price.

i.e. $s_2^* = (2d-c/3d)+1$, which is strictly greater than 1 since $2d > c$. Consequently, if $2d > c$, the pair

$$[(s_1^*, p_1^*), (s_2^*, p_2^*)] = \left(\left\{1, \frac{(2d-c)}{9d}\left[4d+\frac{(2d-c)}{3}+c\right]\right\},\right.$$
$$\left.\left\{1+\frac{(2d-c)}{3d}, \frac{(2d-c)}{9d}\left[2d-\frac{(2d-c)}{3}-c\right]\right\}\right)$$

is the unique perfect equilibrium of the two-stages game.

Thus we see that under vertical product differentiation, there always exists a stable price and location market outcome. In some cases (i.e. $2d \leqslant c$), it is impossible for one of the merchants to secure a strictly positive market share, and thus a strictly positive profit, at such an outcome. In all other cases, the perfect equilibrium provides a unique endogenous determination of both prices and product differences.[1]

III. CONCLUSIONS

The foregoing examples suggest that more stability in price and product competition is to be expected under vertical than under horizontal product differentiation. But we do not know whether this conclusion stands up in a more general setting. A sound approach to this question would require us to establish a general non-existence property of a price equilibrium in horizontal product differentiation, and a general existence property of a price equilibrium in vertical product differentiation. Indeed, the first property would be sufficient to preclude the existence of a perfect equilibrium in the former case whereas, on the contrary, the second property would open the door to the analysis of product selection in the latter case. Needless to say, this seems to be a hopeless task, and we should limit ourselves to confirming that the two examples treated in this paper point in the right direction.

Given this caveat, we believe that it is reasonable to extend the above spatial models to cope with nonspecific transportation cost functions. More precisely, let us now assume that the transport cost between s and s' is given by a function t of the distance $|s-s'|$ which is twice continuously differentiable, increasing and convex.[2] In solving existence problems, it is usual to rely on fixed-point arguments. These arguments can be applied when payoff functions are quasi-concave. In oligopoly theory, a sufficient condition to ascertain the quasi-concavity of profit functions is to show the concavity of demand functions. Clearly, in our problem the properties of the demand functions most crucially hinge on function t. In consequence, the issue of the existence of a price equilibrium can be solved via the determination of the class of transportation cost

[1] In a previous paper (Jaskold Gabszewicz and Thisse, 1979), we have considered a model reminiscent of the present analysis. Two firms are assumed to sell substitute products to a population of consumers with identical tastes but different incomes; the identity of tastes implies that all consumers rank the products in the same order, leading to a preference structure similar to model II (vertical differentiation). The description of a perfect equilibrium in this framework has been provided by Shaked and Sutton (1982).

[2] The convexity of the transportation cost function is the counterpart of the concavity of the utility function.

functions giving rise to concave demand functions. Actually, it turns out that the answers are substantially different in models I and II.

Denote by $\bar{x}(p_1, p_2)$ the location of the consumer indifferent between buying from firm 1 at price p_1 and buying from firm 2 at price p_2, i.e. in model I $\bar{x}$ is the solution to

$$p_1 + t(|\bar{x} - s_1|) = p_2 + t(|\bar{x} - s_2|),$$

while in model II $\bar{x}$ is the solution to

$$p_1 + t(s_1 - \bar{x}) = p_2 + t(s_2 - \bar{x}).$$

Let us first consider model I. Assuming that $\bar{x}$ is between s_1 and s_2, we can show by a standard calculation that[1]

$$\frac{d^2\bar{x}}{dp_1^2} = -\frac{t''(\bar{x} - s_1) - t''(s_2 - \bar{x})}{[t'(\bar{x} - s_1) + t'(s_2 - \bar{x})]^3}.$$

As the demand to firm 1 is $\mu_1(p_1, p_2) = \bar{x}(p_1, p_2)$, the concavity of μ_1 in p_1 is equivalent to the concavity of $\bar{x}$ in p_1. Accordingly $d^2\bar{x}/dp_1^2$ should be non-positive for all prices p_1. Let $\bar{p}_2$ be fixed and such that $\bar{p}_2 > t(s_1, s_2)$. Then, for

$$p_1' \underset{\text{Def}}{=} \bar{p}_2 + t(s_1, s_2), \quad \text{we get} \quad \bar{x}(p_1', \bar{p}_2) = s_1$$

while, for $p_1'' \underset{\text{Def}}{=} \bar{p}_2 - t(s_1, s_2)$, we have $\bar{x}(p_1'', \bar{p}_2) = s_2$. Consequently, it must be that

$$\left.\frac{d^2\bar{x}}{dp_1^2}\right|_{(p_1', \bar{p}_2)} = -\left.\frac{d^2\bar{x}}{dp_1^2}\right|_{(p_1'', \bar{p}_2)}.$$

This means that, unless $d^2\bar{x}/dp_1^2$ equals zero on the interval $[p_1', p_1'']$, $d^2\bar{x}/dp_1^2$ must change its sign on this interval so that $\bar{x}$ cannot be concave in p_1.

Let us now come to model II. It is routine to show that

$$\frac{d^2\bar{x}}{dp_1^2} = \frac{t''(s_1 - \bar{x}) - t''(s_2 - \bar{x})}{[t'(s_1 - \bar{x}) - t'(s_2 - \bar{x})]^3}.$$

Again, the concavity of firm 1's demand in p_1 amounts to the concavity of $\bar{x}$ in p_1. However, unlike model I, $d^2\bar{x}/dp_1^2$ can be nonpositive for all prices p_1. A sufficient condition is that t'' is a non-increasing function of distance. In other words, demand functions are concave in model II when the transportation cost function is not 'too' convex.

When models I and II are modified to account for general transport costs, the following conclusions thus emerge. First, concavity of demands almost never holds in model I. Of course, concavity is only a sufficient condition for the existence of a price equilibrium. Nevertheless, as illustrated in our first example, a significant departure from concavity often yields non-existence. As a result, stability in model I seems very problematic. By contrast, in model II we have identified a whole class of transportation cost functions ensuring the concavity of demands and, thereby, the existence of a price equilibrium. Of course, this is

[1] Notice that $\bar{x}$ may not be differentiable at some points of the price domain. In this case, the argument must be developed in terms of l.h.s. or r.h.s. derivatives.

not enough to imply the existence of a perfect equilibrium. But, at the very least, this gives some hope to have stability in model II.[1]

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APPENDIX

Proposition 1

In model I there exists no price equilibrium when

$$2a \leqslant \min\left(\frac{1}{3}, \frac{\sqrt{c}}{2\sqrt{d}+\sqrt{c}}\right). \quad \text{(A 1)}$$

Proof

Assume that (p_1^*, p_2^*) is a price equilibrium. Three cases may arise. In the first one, (p_1^*, p_2^*) belongs to the domain

$$\mathscr{D}_1 = \{(p_1, p_2);\ \tfrac{1}{2} - a \leqslant \bar{x}(p_1, p_2) \leqslant \tfrac{1}{2} + a\}.$$

In $\mathscr{D}_1$, $\bar{x}(p_1, p_2)$ is given by

$$\frac{p_2 - p_1 + 2ad + c}{4ad + 2c},$$

so that

$$P_1(p_1, p_2) = p_1 \frac{p_2 - p_1 + 2ad + c}{4ad + 2c}$$

[1] It is also worth noting that other models confirm our results. In a general model of horizontal product differentiation, MacLeod (1985) has shown that no price equilibrium exists when firms are close to each other and when marginal production costs fall with output. Furthermore, Jaskold Gabszewicz *et al.* (1981) have established the existence of a price equilibrium in a model of vertical product differentiation for a class of utility functions.

and
$$P_2(p_1,p_2) = p_2\frac{p_1-p_2+2ad+c}{4ad+2c}.$$

Some simple calculations then show that $p_1^* = p_2^* = 2ad+c$, while

$$P_1(p_1^*,p_2^*) = ad+c/2.$$

Given p_2^*, we now study the best reply $\bar{p}_1$ of firm 1 in

$$\mathscr{D}_{11} = \{p_1;\ 0 \leqslant \bar{x}(p_1,p_2^*) < \tfrac{1}{2}-a\}$$

with
$$\bar{x}(p_1,p_2^*) = \frac{p_2^*-p_1+2ad+2ac}{4ad},$$

and in
$$\mathscr{D}_{12} = \{p;\ \tfrac{1}{2}+a < \bar{x}(p_1,p_2^*) \leqslant 1\}$$

with
$$\bar{x}(p_1,p_2^*) = \frac{p_2^*-p_1+2ad-2ac}{4ad}$$

respectively. Consider first the case of $\mathscr{D}_{11}$. The maximum of $p_1\bar{x}(p_1,p_2^*)$ over $[0,\infty[$ is reached at $\tilde{p}_1 = 2ad+ac+c/2$. We easily see that $\bar{x}(\tilde{p}_1,p_2^*) > \tfrac{1}{2}-a$. It then follows that there is no best reply against p_2^* in $\mathscr{D}_{11}$. Let us now come to the case of $\mathscr{D}_{12}$. Given (A 1), it is easy to check that the maximum of $P_1(p_1,p_2^*)$ on $\mathscr{D}_{12}$ is given by the price $\bar{p}_1$ for which $\bar{x}(p_1,p_2^*) = 1$, i.e. $\bar{p}_1 = c(1-2a)$. Using (A 1) again, we obtain $P_1(\bar{p}_1,p_2^*) > P_1(p_1^*,p_2^*)$. Hence $\mathscr{D}_1$ contains no price equilibrium. In the second case, (p_1^*,p_2^*) belongs to

$$\mathscr{D}_2 = \{(p_1,p_2);\ \tfrac{1}{2}+a < \bar{x}(p_1,p_2) \leqslant 1\}.$$

In $\mathscr{D}_2$ we have

$$\bar{x}(p_1,p_2) = \frac{p_2-p_1+2ad-2ac}{4ad},$$

which implies
$$P_1(p_1,p_2) = p_1\frac{p_2-p_1+2ad-2ac}{4ad},$$

and
$$P_2(p_1,p_2) = p_2\frac{p_1-p_2+2ad+2ac}{4ad}.$$

It is easy to show that the solutions $\hat{p}_1$ and $\hat{p}_2$ of the first-order conditions $dP_1/dp_1 = 0$ and $dP_2/dp_2 = 0$ are such that $\bar{x}(\hat{p}_1,\hat{p}_2) > \tfrac{1}{2}+a$. Thus (p_1^*,p_2^*) does not belong to the interior of $\mathscr{D}_2$ and must therefore satisfy $\bar{x}(p_1^*,p_2^*) = 1$. But then p_2^* is not the best reply of firm 2 against p_1^* since $P_2(p_1^*,p_2^*) = 0$. Consequently, no price equilibrium belongs to $\mathscr{D}_2$. In the third case we have

$$(p_1^*,p_2^*) \in \mathscr{D}_3 = \{(p_1,p_2);\ 0 \leqslant \bar{x}(p_1,p_2) < \tfrac{1}{2}-a\}.$$

An argument similar to the above one in which the indices 1 and 2 are permuted shows that $\mathscr{D}_3$ contains no equilibrium. □

Proposition 2

In model II there is a unique price equilibrium given by

$$p_1^*(s_1, s_2) = (s_2 - s_1)\frac{d(s_1+s_2+2)+c}{3}$$

$$p_2^*(s_1, s_2) = (s_2 - s_1)\frac{d(4-s_1-s_2)-c}{3}$$

when $$\frac{c}{d} < 4 - s_1 - s_2 \qquad \text{(A 2)}$$

and by $$p_1^*(s_1, s_2) = (s_2 - s_1)[d(s_1+s_2-2)+c]$$

$$p_2^*(s_1, s_2) = 0$$

when $$\frac{c}{d} \geq 4 - s_1 - s_2. \qquad \text{(A 3)}$$

Proof

Let (p_1^*, p_2^*) be a price equilibrium. (Since the profit functions are quasi-concave, we know that such an equilibrium exists.) Assume first that (A 2) holds. Over the domain

$$\mathscr{D}_1 = \{(p_1, p_2);\ \mu_1(p_1, p_2) > 0 \quad \text{and} \quad \mu_2(p_1, p_2) > 0\},$$

the demand functions are given by

$$\mu_1(p_1, p_2) = \frac{p_2 - p_1 + c(s_2 - s_1) + d(s_2^2 - s_1^2)}{2d(s_2 - s_1)}$$

and $$\mu_2(p_1, p_2) = \frac{p_1 - p_2 + 2d(s_2 - s_1) - c(s_2 - s_1) - d(s_2^2 - s_1^2)}{2d(s_2 - s_1)}.$$

If $(p_1^*, p_2^*) \in \mathscr{D}_1$, then p_1^* and p_2^* must satisfy the first-order conditions

$$\frac{dP_i}{dp_i} = 0, \quad i = 1, 2.$$

Accordingly, $$p_1^*(s_1, s_2) = (s_2 - s_1)\frac{d(s_1+s_2+2)+c}{3}$$

and $$p_2^*(s_1, s_2) = (s_2 - s_1)\frac{d(4-s_1-s_2)-c}{3}.$$

Some simple calculations show that $p_2^* > 0$ and $(p_1^*, p_2^*) \in \mathscr{D}_1$ iff (A 2) holds. For the above prices to be the equilibrium prices under (A 2), it remains to prove that $P_i(p_i^*, p_j^*) > P_i(\bar{p}_i, p_j^*)$, where $\bar{p}_i$ is a best reply of firm i against p_j^* in the domain

$$\mathscr{D}_i = \{p_i;\ \mu_j(p_i, p_j^*) = 0\}.$$

Let $i = 1$. Then

$$\bar{p}_1 = \tfrac{2}{3}(s_2 - s_1)[d(s_1+s_2-1)+c].$$

Using (A 2), some simple calculations yield that $P_1(p_1^*, p_2^*) > P_1(\bar{p}_1, p_2^*)$. Furthermore, due to (A 2), $\mathscr{D}_2$ is empty. Assume now that (A 3) is satisfied. Then, it follows from the above that (p_1^*, p_2^*) must belong to

$$\mathscr{D}_2 = \{(p_1, p_2); \mu_1(p_1, p_2) = 1\}.$$

That p_2^* must be equal to zero follows from the fact that, otherwise, firm 2 could decrease its price and capture a strictly positive market share. Let

$$p_1^L = (s_2 - s_1)[d(s_1 + s_2 - 2) + c]$$

be the solution to $\mu_1(p_1, 0) = 1$. Clearly, p_1^L dominates any price $p_1 < p_1^L$. Moreover, it can be shown that $dP_1/dp_1 < 0$ for $p_1 > p_1^L$ if (A 3) is satisfied. Consequently, we have $p_1^* = p_1^L$. □